

# AUSTERE SUBMANIFOLDS IN $\mathbb{C}P^n$

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ABSTRACT. We determine conditions under which an arbitrary submanifold  $M \subset \mathbb{C}P^n$  is *austere* with respect to Stenzel's Ricci-flat Kähler metric on  $T\mathbb{C}P^n$ , i.e., the normal bundle of  $M$  is special Lagrangian. We also classify austere surfaces in  $\mathbb{C}P^n$ .

## 1. INTRODUCTION

Special Lagrangian submanifolds were introduced in 1982 by Harvey and Lawson in their seminal paper [2]. They studied them in the more general context of calibrated submanifolds, which are a special class of minimal submanifolds. Calibrated submanifolds, in particular special Lagrangian submanifolds, play an important role in mirror symmetry and they have been the object of extensive study lately. Most of the earlier research has focused on special Lagrangian submanifolds in  $\mathbb{C}^n$  (for this, see Joyce [5] and the extensive references contained therein).

Let  $\mathcal{X}^n$  be a Calabi-Yau manifold with Kähler form  $\omega$  and holomorphic volume form  $\Omega$ . Recall that an oriented real  $n$ -dimensional submanifold  $L$  is *special Lagrangian* if it is calibrated by  $\operatorname{Re} \Omega$ . Harvey and Lawson showed that  $L$  is special Lagrangian if and only if  $\omega|_L \equiv 0$  and  $\operatorname{Im} \Omega|_L \equiv 0$ . (The same is true if we replace  $\Omega$  by  $e^{i\theta}\Omega$ , in which case  $L$  is said to be special Lagrangian *with phase*  $e^{i\theta}$ .) In the same paper [2], Harvey and Lawson exhibited a construction of special Lagrangian submanifold using bundles. More specifically, they showed that the conormal bundle  $N^*M$  of an immersed submanifold  $M^k \subset \mathbb{R}^n$  is special Lagrangian in  $\mathbb{C}^n \cong T^*\mathbb{R}^n$  if and only if  $M^k$  is *austere* in  $\mathbb{R}^n$ , i.e. the second fundamental form of  $M$  in any normal direction has its eigenvalues symmetrically arranged around 0. This is equivalent to saying that all the odd-degree symmetric polynomials in the eigenvalues of the second fundamental form vanish identically. Note that the austere condition in general is stronger than the minimal one.

In the early 1990s, Stenzel [9, 10] showed that the cotangent bundle of any rank one symmetric space can be endowed with a Ricci-flat metric, which is now called a *Stenzel metric*. Particular cases of Stenzel metrics are the ones discovered initially by Eguchi-Hanson on the cotangent bundle of the sphere  $S^2$  and by Candela-de la Ossa on  $T^*S^3$ . In [6] Karagiannis and Min-Oo generalized Harvey and Lawson's construction to the cotangent bundle of  $S^n$  carrying the Stenzel Ricci-flat metric. More specifically, they showed that the conormal bundle over an immersed submanifold  $M \subset S^n$  is special Lagrangian with respect to some phase if and only if in any normal direction, all the odd-degree symmetric polynomial in the eigenvalues of the second fundamental form vanish identically. In other words, this is the same austere condition as Harvey and Lawson found in [2] for  $\mathbb{C}^n$ . This is perhaps surprising, since the complex structure on  $T^*S^n$  is not the standard one as in the case of  $\mathbb{C}^n \cong T^*\mathbb{R}^n$ , but instead is obtained by identifying it with a complex hyperquadric in  $\mathbb{C}^{n+1}$ .

In this paper, we further generalize the Harvey and Lawson construction to the case of  $T^*\mathbb{C}P^n$ , the cotangent bundle of complex projective space. We will define  $M \subset \mathbb{C}P^n$  to be *austere* if its conormal bundle  $N^*M$  is special Lagrangian in  $T^*\mathbb{C}P^n$ , with respect to the Stenzel metric. (In fact, we will work on the normal bundle  $NM$ , using the standard metric on  $\mathbb{C}P^n$  to identify  $T^*\mathbb{C}P^n$  with  $T\mathbb{C}P^n$ .)

We now give a brief description of the contents of the paper. In section 2, we define a mapping  $\Phi$  that identifies  $T\mathbb{C}P^n$  with a Stein submanifold of  $\mathbb{C}P^n \times \mathbb{C}P^n$  that is the complement of a complex quadric, and we calculate the differential of restriction of  $\Phi$  to  $NM$  using moving frames. At the end of this section, we prove that if  $M \subset \mathbb{C}P^n$  is an arbitrary immersed submanifold, then  $NM$  is a Lagrangian submanifold of  $T\mathbb{C}P^n$ , with respect to the Stenzel Kähler form (see Prop. 1). In section 3, we determine the conditions under which an immersed submanifold  $M \subset \mathbb{C}P^n$  is austere (see Theorem 4). A particular case of this result is that if  $M \subset \mathbb{C}P^n$  is an arbitrary holomorphic submanifold, then  $M$  is austere (see Corollary 2). Finally, in section 4, we classify the austere surfaces in  $\mathbb{C}P^n$ , showing that they must be either holomorphic curves or totally geodesic (see Theorem 7).

Before proceeding with the calculations leading up to Proposition 1, we need to make a few remarks:

- (1) As indicated above, our calculations will be made using moving frames. Because such frames are easier to differentiate when the individual frame vectors take value in a fixed space, we will do most calculations on  $\widehat{M} \subset S^{2n+1} \subset \mathbb{C}^{n+1}$ , which is the inverse image of  $M$  under the Hopf fibration  $\pi : S^{2n+1} \rightarrow \mathbb{C}P^n$ . (We also use  $\pi$  to denote the projectivization map from  $\mathbb{C}^{n+1}/\{0\}$  to  $\mathbb{C}P^n$ .) Similarly, we will do calculations on  $NM \subset T\mathbb{C}P^n$  by regarding  $T\mathbb{C}P^n$  as a quotient space and working on the inverse image. In more detail, let

$$B = \{(\zeta, \xi) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \mid \zeta \neq 0, \xi \cdot \bar{\zeta} = 0\},$$

wherein the dot product is  $\mathbb{C}$ -bilinear. Recall that  $T_L\mathbb{C}P^n$  is canonically defined as the set of  $\mathbb{C}$ -linear maps  $f$  from the one-dimensional subspace  $L \subset \mathbb{C}^{n+1}$  to the quotient space  $\mathbb{C}^{n+1}/L$ . If  $f$  is such a map and  $\zeta$  is a nonzero point on  $L$ , there is a unique  $\xi$  which projects to  $f(\zeta)$  in the quotient space and satisfies  $\xi \cdot \bar{\zeta} = 0$ . Thus,  $T\mathbb{C}P^n \equiv B/\mathbb{C}^*$ , where the  $\mathbb{C}^*$  action on  $B$  is  $(\zeta, \xi) \mapsto (\lambda\zeta, \lambda\xi)$  for  $\lambda \in \mathbb{C}^*$ .

- (2) Although the cotangent bundle of any complex manifold  $\mathcal{X}$  has a standard complex structure (obtained by identifying it with the bundle of  $(1, 0)$ -forms), the complex structure underlying the Stenzel metric on  $T^*\mathbb{C}P^n$  is not the standard one. For example, under the mapping  $\Phi$  the image of the zero section is a totally real submanifold.
- (3) For an arbitrary submanifold  $M \subset \mathbb{C}P^n$  and  $p \in M$ , let  $\mathcal{H}_p$  and  $\mathcal{N}_p$  be maximal  $J$ -invariant subspaces of  $T_pM$  and  $N_pM$  respectively. We'll assume that  $\mathcal{H} = \bigcup_{p \in M} \mathcal{H}_p$  and  $\mathcal{N} = \bigcup_{p \in M} \mathcal{N}_p$  are smooth sub-bundles of  $TM$  and  $NM$ , and let  $\mathcal{D}$  and  $\mathcal{E}$  be their respective orthogonal complements. Thus, we have an orthogonal splitting

$$T\mathbb{C}P^n|_M = TM \oplus NM = (\mathcal{H} \oplus \mathcal{D}) \oplus (\mathcal{E} \oplus \mathcal{N}) \quad (1)$$

such that  $\mathcal{D} \oplus \mathcal{E}$  is  $J$ -invariant and  $\text{rk } \mathcal{D} = \text{rk } \mathcal{E} \leq n$ . As we will see, the austere condition along a fiber of  $NM$  depends on how the corresponding normal vector splits into components in  $\mathcal{E}$  and  $\mathcal{N}$ .

We gratefully acknowledge the comments and helpful discussions with the following mathematicians during our work on this paper: Henri Anciaux, Ronan Conlon, Spiro Karigiannis, JaeHyoun Lee, Conan Leung, Paul-Andi Nagy, and Pat Ryan.

## 2. LAGRANGIAN SUBMANIFOLDS VIA NORMAL BUNDLES

As above, let  $\widehat{M} = \pi^{-1}(M)$  where  $\pi : S^{2n+1} \rightarrow \mathbb{C}P^n$  is the Hopf fibration. (We will regard points of  $S^{2n+1}$ , as well as tangent vectors to the sphere, as vectors in  $\mathbb{C}^{n+1}$ .) Let  $\mathcal{F}$  denote the bundle of adapted orthonormal frames along  $\widehat{M}$ , where the fiber  $\mathcal{F}_z$  at a point  $z \in \widehat{M}$  consists of oriented orthonormal frames  $(iz, e_1, \dots, e_{2n})$  such that  $e_1, \dots, e_k$  are tangent to  $\widehat{M}$ . Note that the vector  $iz$

is tangent to the fiber of  $\pi$  through  $\mathbf{z}$ , while  $e_1, \dots, e_{2n}$  are orthogonal to the fiber, and so are called *horizontal* vectors. The fibers of  $\pi : \widehat{M} \rightarrow M$  are, of course, orbits of the  $S^1$  action  $\mathbf{z} \mapsto e^{i\theta}\mathbf{z}$ . This action extends to  $\mathcal{F}$ , simply by simultaneously multiplying all frame vectors by  $e^{i\theta}$ , and preserves horizontality.

Given any  $\mathbf{z} \in \widehat{M}$  and any normal vector  $\mathbf{n} \in N_{\mathbf{z}}\widehat{M}$  (necessarily horizontal), there exists an adapted frame at  $\mathbf{z}$  such that  $\mathbf{n} = te_{2n}$  for some  $t \in \mathbb{R}$ . Thus, we define a submersion

$$\varrho : ((\mathbf{z}, e_1, \dots, e_{2n}), t) \mapsto te_{2n} \in T_{\mathbf{z}}S^{2n+1}$$

from  $\mathcal{F} \times \mathbb{R}$  to  $N\widehat{M}$ . Because any normal vector  $\nu \in T_{\pi(\mathbf{z})}M$  has a horizontal lift in  $T_{\mathbf{z}}\widehat{M}$ , there is also a submersion  $\Pi : N\widehat{M} \rightarrow NM$  defined by  $\mathbf{n} \mapsto \pi_*\mathbf{n}$ . We define yet another submersion

$$\rho : ((\mathbf{z}, e_1, \dots, e_{2n}), t) \mapsto (\mathbf{z}, te_{2n}) \in B$$

that lifts the natural inclusion  $\iota : NM \rightarrow T\mathbb{C}P^n$ . In other words, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F} \times \mathbb{R} & \xrightarrow{\rho} & B \\ \varrho \downarrow & & \downarrow \\ N\widehat{M} & & \\ \Pi \downarrow & & \downarrow \\ NM & \xrightarrow{\iota} & T\mathbb{C}P^n \end{array}$$

where the vertical map at right is quotient by the  $\mathbb{C}^*$  action.

To generate an embedding of  $NM$  as a submanifold in the Stein manifold  $\mathcal{M}$ , we will compose  $\rho$  with a map  $\widehat{\Phi} : B \rightarrow \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$  defined by

$$\widehat{\Phi}(\zeta, \xi) = \left( (\cosh \mu)\zeta + i \left( \frac{\sinh \mu}{\mu} \right) \xi; (\cosh \mu)\bar{\zeta} + i \left( \frac{\sinh \mu}{\mu} \right) \bar{\xi} \right), \quad \mu = \frac{|\xi|}{|\zeta|}. \quad (2)$$

(This is adapted from the work of Szöke [11].)

[**Notation:** We will write vectors in  $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$  as an ordered pair of vectors in  $\mathbb{C}^{n+1}$  separated by a semicolon.]

It is easy to check that, relative to the  $\mathbb{C}^*$  action on  $B$  and projectivization on each factor in  $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ ,  $\widehat{\Phi}$  covers a well-defined mapping  $\Phi : T\mathbb{C}P^n \rightarrow \mathbb{C}P^n \times \mathbb{C}P^n$  that identifies  $T\mathbb{C}P^n$  bijectively with  $\mathcal{M}$ . We will compute the differential of the composition of  $\Phi$  with the inclusion *imath* by computing the differential of the composition of maps along the top edge of the diagram

$$\begin{array}{ccccccc} \mathcal{F} \times \mathbb{R} & \xrightarrow{\rho} & B & \xrightarrow{\widehat{\Phi}} & \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} & & \\ \varrho \downarrow & & \downarrow & & \downarrow & \searrow \widehat{\mathfrak{A}} & \\ N\widehat{M} & & & & & & \\ \Pi \downarrow & & \downarrow & & \downarrow & & \\ NM & \xrightarrow{\iota} & T\mathbb{C}P^n & \xrightarrow{\Phi} & \mathbb{C}P^n \times \mathbb{C}P^n & \xrightarrow{\mathfrak{A}} & \mathbb{C}^{2n} \end{array}$$

in which  $\mathfrak{A}$  denotes an affine coordinate chart (to be specified below) and  $\widehat{\mathfrak{A}}$  is the corresponding map in terms of homogeneous coordinates. As we will see at the end of the next section, the differential along the top edge annihilates vectors that are tangent to the fibers of the map  $\Pi \circ \varrho$ .

**2.1. Computing the Differential.** The full oriented orthonormal frame bundle of  $S^{2n+1}$  may be identified with the special orthogonal group  $SO(2n+2)$ , by taking the basepoint and the frame vectors as successive rows of an orthogonal matrix. However, in what follows we will mostly regard

these vectors as belonging in  $\mathbb{C}^{n+1}$ . When we wish to identify them with vectors in  $\mathbb{R}^{2n+2}$ , we use the following convention (and notation): for  $\mathbf{v} \in \mathbb{C}^N$ , define

$$\widehat{\mathbf{v}} := (\operatorname{Re} \mathbf{v}; \operatorname{Im} \mathbf{v}) \in \mathbb{R}^{2N}.$$

In terms of this notation, we will assume that

$$\begin{pmatrix} \widehat{e_0} \\ \widehat{ie_0} \\ \widehat{e_1} \\ \vdots \\ \widehat{e_{2n}} \end{pmatrix} \in SO(2n+2). \quad (3)$$

(For the sake of consistency below, we have renamed the basepoint  $\mathbf{z}$  as the unit vector  $e_0$ .) This embedding of  $\mathcal{F}$  as a subgroup of  $SO(2n+2)$  endows it with a natural coframe made up of components of the Maurer-Cartan form. Among these are real-valued 1-forms

$$\omega^0, \omega^\alpha, \psi_{2n}^\alpha, \psi_{2n}^\nu, \psi_{2n}^0 \quad (4)$$

that are defined by expanding the exterior derivatives of the frame vectors in terms of the basis (over  $\mathbb{R}$ ) provided by the frame:

$$de_0 = e_\alpha \omega^\alpha + ie_0 \omega^0, \quad de_{2n} = e_\alpha \psi_{2n}^\alpha + e_\mu \psi_{2n}^\mu + ie_0 \psi_{2n}^0. \quad (5)$$

[**Notation:** Here and in the sequel we use summation convention with index ranges  $1 \leq \alpha, \beta \leq k < \mu, \nu < 2n$ . In (5) each term is a vector-valued 1-form, i.e., a section of  $\mathbb{C}^{n+1} \otimes T^*\mathcal{F}$ . While we omit the tensor product symbol in (5) and write the vector factor on the left, it will occasionally be convenient to write the vector on the right, as in (7), (8) below.]

Any 1-form on  $\mathcal{F} \times \mathbb{R}$  that is semibasic for  $\varrho$  is in the span of  $dt$  and the 1-forms (4). These 1-forms are not linearly independent, since

$$\psi_{2n}^0 = \langle ie_0, de_{2n} \rangle = -\langle e_0, d(ie_{2n}) \rangle = \langle de_0, ie_{2n} \rangle,$$

where  $\langle \cdot, \cdot \rangle$  denotes the real-valued Euclidean inner product on  $\mathbb{C}^{n+1}$ . Thus, if we set  $r_\alpha = \langle ie_{2n}, e_\alpha \rangle$ , then  $\psi_{2n}^0 = r_\alpha \omega^\alpha$ . (Note that if  $M$  is a holomorphic submanifold then all  $r_\alpha = 0$ .) As well, differentiating the first equation in (5) shows that

$$\psi_{2n}^\alpha = r_\alpha \omega^0 - h_{\alpha\beta} \omega^\beta,$$

and it is easy to check that  $h_{\alpha\beta} = \underline{e}_{2n} \cdot \text{II}(\underline{e}_\alpha, \underline{e}_\beta)$ , where  $\text{II}$  is the second fundamental form of  $M$ .

We substitute the above formulas (5) into

$$\begin{aligned} d(\widehat{\Phi} \circ \rho) &= (\cosh t de_0 + i \sinh t de_{2n} + (\sinh t e_0 + i \cosh t e_{2n}) dt; \\ &\quad \cosh t d\overline{e_0} + i \sinh t d\overline{e_{2n}} + (\sinh t \overline{e_0} + i \cosh t \overline{e_{2n}}) dt), \end{aligned}$$

giving

$$\begin{aligned} d(\widehat{\Phi} \circ \rho) &= \cosh t [\omega^0, \omega^\alpha, \psi_{2n}^\mu, dt] \\ &\otimes \left[ \begin{pmatrix} i & 0 & 0 & 0 \\ -\tau r_\alpha & \delta_{\alpha\beta} - i\tau h_{\alpha\beta} & 0 & 0 \\ 0 & 0 & i\tau \delta_{\mu\nu} & 0 \\ \tau & 0 & 0 & i \end{pmatrix} \begin{bmatrix} e_0 \\ e_\beta \\ e_\nu \\ e_{2n} \end{bmatrix}; \begin{pmatrix} -i & 0 & 0 & 0 \\ \tau r_\alpha & \delta_{\alpha\beta} - i\tau h_{\alpha\beta} & 0 & 0 \\ 0 & 0 & i\tau \delta_{\mu\nu} & 0 \\ \tau & 0 & 0 & i \end{pmatrix} \begin{bmatrix} \overline{e_0} \\ \overline{e_\beta} \\ \overline{e_\nu} \\ \overline{e_{2n}} \end{bmatrix} \right]. \quad (6) \end{aligned}$$

(The matrices are partitioned into block form, with column and row widths 1,  $k$ ,  $n - k - 1$  and 1; we have also introduced the abbreviation  $\tau = \tanh t$ .)

We will use the isometry group  $U(n+1)$  to simplify these matrices, by moving any horizontal vector in  $N\widehat{M}$  into a standard position. For, given any point  $\mathbf{z} \in \widehat{M}$  and a horizontal vector  $\mathbf{h} \in T_{\mathbf{z}}S^{2n+1}$ , we can arrange that

$$\mathbf{z} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} 0 & \dots & 0 & \mathbf{i} \end{bmatrix}.$$

Thus, from now on we will assume that  $e_0 = E_0$  and  $e_{2n} = \mathbf{i}E_n$ , where  $E_0, \dots, E_n$  denote the elementary basis row vectors of  $\mathbb{C}^{n+1}$ . Then

$$\widehat{\Phi} \circ \rho((e_0, \dots, e_{2n}), t) = (\cosh t, 0, \dots, 0, -\sinh t; \cosh t, 0, \dots, 0, \sinh t).$$

Let  $o \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$  be the point on the right-hand side. For any  $t$  this is in the domain of the map

$$\widehat{\mathfrak{A}} : (z_0, \dots, z_n; w_0, \dots, w_n) \mapsto \left( \frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}, \frac{w_1}{w_0}, \dots, \frac{w_n}{w_0} \right).$$

which covers the affine coordinate chart  $\mathfrak{A}$ . We compute the differential of  $\widehat{\mathfrak{A}}$ , and evaluate at  $o$ :

$$d\widehat{\mathfrak{A}}_o = \operatorname{sech} t (dz_1, \dots, dz_{n-1}, dz_n + \tau dz_0; dw_1, \dots, dw_{n-1}, dw_n - \tau dw_0).$$

To compute the differential of  $\widehat{\mathfrak{A}} \circ \widehat{\Phi} \circ \rho$ , we apply the  $\mathbb{C}$ -linear map  $d\widehat{\mathfrak{A}}_o$  to the vector factors in the tensor product in (6). For example,

$$d\widehat{\mathfrak{A}}_o(e_\alpha; 0) = (\widetilde{e}_\alpha; 0), \quad d\widehat{\mathfrak{A}}_o(e_{2n}; 0) = (\mathbf{i}\widetilde{E}_n; 0), \quad d\widehat{\mathfrak{A}}_o(e_0; 0) = (\tau\widetilde{E}_n; 0), \quad d\widehat{\mathfrak{A}}_o(0; e_0) = (0, -\tau\widetilde{E}_n),$$

where the  $\sim$  indicates the result of deleting the first entry from a row vector in  $\mathbb{C}^{n+1}$ , and we use the fact that, because the vectors  $e_\alpha$  and  $e_\nu$  are horizontal at  $\mathbf{z} = E_0$ , their first entries are zero. Computing in this way, we obtain

$$d(\widehat{\mathfrak{A}} \circ \widehat{\Phi} \circ \rho) = [\omega^0, \omega^\alpha, \psi_{2n}^\mu, dt] \otimes \begin{pmatrix} \mathbf{i}\tau\widetilde{E}_n & ; & \mathbf{i}\tau\widetilde{E}_n \\ \widetilde{e}_\alpha - \mathbf{i}\tau h_{\alpha\beta}\widetilde{e}_\beta - \tau^2 r_\alpha \widetilde{E}_n & ; & \widetilde{e}_\alpha - \mathbf{i}\tau h_{\alpha\beta}\widetilde{e}_\beta - \tau^2 r_\alpha \widetilde{E}_n \\ \mathbf{i}\tau\widetilde{e}_\mu & ; & \mathbf{i}\tau\widetilde{e}_\mu \\ (\tau^2 - 1)\widetilde{E}_n & ; & (1 - \tau^2)\widetilde{E}_n \end{pmatrix}. \quad (7)$$

While the 1-forms  $\omega^0, \omega^\alpha, \psi_{2n}^\mu$  and  $dt$  are linearly independent and span the semibasic 1-forms for  $\varrho$ , it is evident from (5) that  $\omega^0$  is not semibasic for  $\Pi \circ \varrho$ . In fact, if  $\mathbf{v}$  is the vector field on  $\mathcal{F} \times \mathbb{R}$  that generates the  $S^1$  action under which  $NM = N\widehat{M}/S^1$ , then

$$\mathbf{v} \lrcorner \omega^0 = 1, \quad \mathbf{v} \lrcorner \omega^\alpha = 0, \quad \mathbf{v} \lrcorner \psi_{2n}^\mu = r_\mu := \langle \mathbf{i}e_{2n}, e_\mu \rangle.$$

Using these formulas, it is easy to check that  $\mathbf{v}$  is in the kernel of the differential (7). In fact, the  $r_\mu$  give the coefficients under which the top row of the matrix is a linear combination of the third set of rows below it.

Thus, in terms of the diagram (6),  $\widehat{\mathfrak{A}} \circ \widehat{\Phi} \circ \rho$  covers a well-defined map  $\mathfrak{A} \circ \Phi \circ \iota$  from  $NM$  to  $\mathbb{C}^{2n}$ . Matters being so, we will expand the differential just in terms of the 1-forms  $\omega^\alpha, dt$  and  $\widetilde{\psi}_{2n}^\mu := \psi_{2n}^\mu - r_\mu \omega^0$ , as

$$d(\widehat{\mathfrak{A}} \circ \widehat{\Phi} \circ \rho) = [\omega^\alpha, \widetilde{\psi}_{2n}^\mu, dt] \otimes \begin{pmatrix} \widetilde{e}_\alpha - \mathbf{i}\tau h_{\alpha\beta}\widetilde{e}_\beta - \tau^2 r_\alpha \widetilde{E}_n & ; & \widetilde{e}_\alpha - \mathbf{i}\tau h_{\alpha\beta}\widetilde{e}_\beta - \tau^2 r_\alpha \widetilde{E}_n \\ \mathbf{i}\tau\widetilde{e}_\mu & ; & \mathbf{i}\tau\widetilde{e}_\mu \\ (\tau^2 - 1)\widetilde{E}_n & ; & (1 - \tau^2)\widetilde{E}_n \end{pmatrix}. \quad (8)$$

**2.2. The Stenzel Kähler Form.** A convenient explicit description of the Stenzel metric in local coordinates on  $\mathbb{C}P^n \times \mathbb{C}P^n$  is given by T-C. Lee [7]; we briefly reproduce it here for the sake of the calculations in §2.3. Lee defines two functions on  $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ ,

$$\mathcal{A} = \sum_{j,k=0}^n |z_j w_k|^2, \quad \mathcal{B} = \mathbf{z} \cdot \mathbf{w} = \sum_{j=0}^n z_j w_j,$$

which are homogeneous of degree 4 and 2 respectively; then the exhaustion function  $\mathcal{N} = \mathcal{A}/|\mathcal{B}|^2$  is well-defined and smooth on  $\mathcal{M}$ . The Kähler potential  $f(\mathcal{N})$  for the Stenzel metric satisfies  $f' = \mathcal{N}^{-1/2}$ . Using this, we calculate the Kähler form in terms of affine coordinates  $Z_a = z_a/z_0$  and  $W_a = w_a/w_0$ , where we take  $1 \leq a, b \leq n$ .

To start with,

$$\bar{\partial}\mathcal{A} = (1 + |W|^2)Z_b d\bar{Z}_b + (1 + |Z|^2)W_b d\bar{W}_b$$

where  $|Z|^2 = \sum_a Z_a \bar{Z}_a$  and similarly; then

$$\begin{aligned} \partial\bar{\partial}\mathcal{A} &= (1 + |W|^2)dZ_b \wedge d\bar{Z}_b + \bar{W}_a Z_b dW_a \wedge d\bar{Z}_b + \bar{Z}_a W_b dZ_a \wedge d\bar{W}_b + (1 + |Z|^2)dW_b \wedge d\bar{W}_b \\ &= (dZ dW) \wedge \begin{pmatrix} (1 + |W|^2)I_n & \bar{Z}^T W \\ \bar{W}^T Z & (1 + |Z|^2)I_n \end{pmatrix} \begin{pmatrix} d\bar{Z}^T \\ d\bar{W}^T \end{pmatrix}, \end{aligned} \quad (9)$$

where we regard  $Z$  and  $W$  as row vectors of length  $n$ ,  $I_n$  is the  $n \times n$  matrix, and  $^T$  indicates transpose. Following Lee, we identify the  $(1,1)$ -form  $\partial\bar{\partial}\mathcal{A}$  with the hermitian matrix in (9) that gives its coefficients with respect to the affine coordinates; similarly, we identify  $\partial\mathcal{A}$  with the row vector  $((1 + |W|^2)\bar{Z}; (1 + |Z|^2)\bar{W})$  of length  $2n$ . With these conventions, we identify  $\partial\bar{\partial}f$  with the hermitian matrix

$$\mathbf{G} = \frac{f'}{|\mathcal{B}|^2} \left[ \partial\bar{\partial}\mathcal{A} - \frac{1}{\mathcal{A}}(\partial\mathcal{A})^T \bar{\partial}\mathcal{A} + \left( \frac{f''}{|\mathcal{B}|^2 f'} + \frac{1}{\mathcal{A}} \right) (\partial\mathcal{A} - (\mathcal{A}/\mathcal{B})\partial\mathcal{B})^T (\bar{\partial}\mathcal{A} - (\mathcal{A}/\bar{\mathcal{B}})\bar{\partial}\bar{\mathcal{B}}) \right].$$

We will only need the value of the metric at the point  $\check{o} = \widehat{\mathfrak{A}}(o)$ , where  $W_n = \tau = \tanh t$ ,  $Z_n = -\tau$  and all other coordinates are zero. At this point, we have

$$\mathbf{G} = \frac{1}{1 - \tau^4} \left[ (1 + \tau^2)I_{2n} + \begin{pmatrix} (q - \tau^2)\mathbf{M} & -q\mathbf{M} \\ -q\mathbf{M} & (q - \tau^2)\mathbf{M} \end{pmatrix} \right], \quad (10)$$

where

$$\mathbf{M} = \widetilde{E_n}^T \widetilde{E_n}, \quad \text{and} \quad q = \frac{2\tau^2}{(1 - \tau^2)^2}.$$

**2.3. Checking Lagrangian-ness.** In this section, we prove:

**Proposition 1.** *If  $M \subset \mathbb{C}P^n$  is an arbitrary submanifold, then  $\Phi(NM)$  is a Lagrangian submanifold of  $\mathcal{M}$ .*

*Proof.* We will identify a real tangent vector  $\mathbf{v} \in T_{\check{o}}\mathcal{M}$  with the row vector in  $\mathbb{C}^{2n}$  given by  $\mathbf{v} \lrcorner (dZ; dW)$ . With this convention, the Stenzel metric  $g$  satisfies

$$g(\mathbf{v}, \mathbf{w}) = 2 \operatorname{Re}(\bar{\mathbf{v}} \mathbf{G} \mathbf{w}^T), \quad (11)$$

and its Kähler form  $\Omega = i\partial\bar{\partial}f$  satisfies

$$\Omega(\mathbf{v}, \mathbf{w}) = -2 \operatorname{Im}(\bar{\mathbf{v}} \mathbf{G} \mathbf{w}^T), \quad \forall \mathbf{v}, \mathbf{w} \in T_{\check{o}}\mathcal{M}, \quad (12)$$

The process of verifying that  $\Omega$  vanishes on the tangent space to  $\Phi(NM)$  is made simpler by decomposing tangent vectors into vertical and horizontal pieces. (By vertical vectors, we mean

those tangent to the images under  $\Phi$  of the fibers of  $T\mathbb{C}P^n$ , and the horizontal vectors are those in the orthogonal complement with respect to  $g$ .) Computing

$$\left. \frac{d}{ds} \right|_{s=0} \widehat{\mathfrak{A}} \circ \widehat{\Phi}(\zeta, \xi + s\eta)$$

when  $\zeta = E_0$ ,  $\xi = itE_n$  and  $\eta \in C^{n+1}$  ranges over all vectors satisfying  $\eta \cdot \bar{\zeta} = 0$ , shows that the space of vertical tangent vectors at  $\check{o}$  consists of all vectors of the form  $(\mathbf{z}; -\bar{\mathbf{z}})$  for  $\mathbf{z} \in \mathbb{C}^n$ . Then, noting the special form of  $\mathbf{G}$  in (10), it is easy to check using (11) that the space of horizontal vectors consists of all vectors of the form  $(\mathbf{z}; \bar{\mathbf{z}})$ . It is also easy to check that  $\Omega(\mathbf{v}, \mathbf{w}) = 0$  whenever  $\mathbf{v}$  and  $\mathbf{w}$  are both vertical or both horizontal.

Equation (8) shows that the tangent space to  $\Phi(NM)$  at  $\check{o}$  is spanned by purely vertical vectors  $(i\check{e}_\mu; i\check{e}_\mu)$  and  $(\check{E}_n; -\check{E}_n)$ , and the ‘mixed’ vectors

$$\mathbf{u}_\alpha := (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n; \check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) - \tau h_{\alpha\beta} (i\check{e}_\beta; i\check{e}_\beta).$$

(Note that the first term is horizontal and the second is vertical.) When evaluating  $\Omega$  on the pairing of  $\mathbf{u}_\alpha$  with a purely vertical vector, we only have to use the horizontal part of  $\mathbf{u}_\alpha$ . For example, we compute using (10) and (12) that

$$\begin{aligned} \Omega(\mathbf{u}_\alpha, (\check{E}_n; -\check{E}_n)) &= \frac{-2}{1-\tau^4} \operatorname{Im} \left[ (1+\tau^2) \left( (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \check{E}_n + (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot (-\check{E}_n) \right) \right. \\ &\quad + (q-\tau^2) \left( (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathbf{M}\check{E}_n + (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathbf{M}(-\check{E}_n) \right) \\ &\quad \left. - q \left( (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathbf{M}(-\check{E}_n) + (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathbf{M}\check{E}_n \right) \right] = 0. \end{aligned}$$

In fact, the terms on each line inside the square brackets cancel out because  $\mathbf{M}\check{E}_n = \check{E}_n$  and because

$$0 = \langle e_\alpha, e_{2n} \rangle = \operatorname{Re}(\check{e}_\alpha \cdot i\check{E}_n),$$

so that  $\check{e}_\alpha \cdot \check{E}_n$  is real (and equal to  $-r_\alpha$ ). Pairing with the other vertical vectors, we get

$$\begin{aligned} \Omega(\mathbf{u}_\alpha, (i\check{e}_\mu; i\check{e}_\mu)) &= \frac{-2}{1-\tau^4} \operatorname{Im} \left[ (1+\tau^2) \left( (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot i\check{e}_\mu + (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot i\check{e}_\mu \right) \right. \\ &\quad + (q-\tau^2) \left( (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot i\mathbf{M}\check{e}_\mu + (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot i\mathbf{M}\check{e}_\mu \right) \\ &\quad \left. - q \left( (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot i\check{e}_\mu + (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot i\check{e}_\mu \right) \right] \end{aligned}$$

Again,  $\check{e}_\mu \cdot \check{E}_n = -r_\mu$  is real, and so  $\mathbf{M}\check{e}_\mu = -r_\mu \check{E}_n$ ; we also note that

$$0 = \langle e_\alpha, e_\mu \rangle = \operatorname{Re}(\check{e}_\alpha \cdot \check{e}_\mu) = \frac{1}{2}(\check{e}_\alpha \cdot \check{e}_\mu + \check{e}_\alpha \cdot \check{e}_\mu).$$

Thus,

$$\Omega(\mathbf{u}_\alpha, (i\check{e}_\mu; i\check{e}_\mu)) = \frac{-4}{1-\tau^4} \left[ (1+\tau^2)\tau^2 r_\alpha r_\mu + (q-\tau^2)(1+\tau^2)r_\alpha r_\mu - q(1+\tau^2)r_\alpha r_\mu \right] = 0.$$

It remains to check that  $\Omega$  is zero on a pair  $\mathbf{u}_\alpha, \mathbf{u}_\beta$  of ‘mixed’ vectors. Let

$$\mathbf{v}_\alpha = (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n; \check{e}_\alpha - \tau^2 r_\alpha \check{E}_n),$$

the horizontal part of  $\mathbf{u}_\alpha$ , and let  $\mathbf{w}_\alpha = (\mathbf{i}\check{e}_\beta; \mathbf{i}\check{e}_\beta)$ . First, we compute that

$$\begin{aligned}\Omega(\mathbf{v}_\alpha, \mathbf{w}_\beta) &= \frac{-2}{1-\tau^4} \operatorname{Im} \left[ (1+\tau^2) \left( (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathbf{i}\check{e}_\beta + (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathbf{i}\check{e}_\beta \right) \right. \\ &\quad \left. + (q-\tau^2) \left( (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathbf{i}M\check{e}_\beta + (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathbf{i}M\check{e}_\beta \right) \right. \\ &\quad \left. - q \left( (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathbf{i}M\check{e}_\beta + (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathbf{i}M\check{e}_\beta \right) \right] \\ &= \frac{-4}{1-\tau^4} \left[ (1+\tau^2)(\delta_{\alpha\beta} + \tau^2 r_\alpha r_\beta) + (q-\tau^2)(1+\tau^2)r_\alpha r_\beta - q(1+\tau^2)r_\alpha r_\beta \right] = \frac{-4}{1-\tau^2} \delta_{\alpha\beta},\end{aligned}$$

where we have used the fact that  $M\mathbf{e}_\beta = -r_\beta \check{E}_n$  and

$$\delta_{\alpha\beta} = \langle e_\alpha, e_\beta \rangle = \operatorname{Re}(\check{e}_\alpha \cdot \check{e}_\beta).$$

Then

$$\begin{aligned}\Omega(\mathbf{u}_\alpha, \mathbf{u}_\beta) &= \Omega(\mathbf{v}_\alpha - \tau h_{\alpha\gamma} \mathbf{w}_\gamma, \mathbf{v}_\beta - \tau h_{\beta\epsilon} \mathbf{w}_\epsilon) \\ &= -\tau h_{\alpha\gamma} \Omega(\mathbf{w}_\gamma, \mathbf{v}_\beta) - \tau h_{\beta\epsilon} \Omega(\mathbf{v}_\alpha, \mathbf{w}_\epsilon) \\ &= \frac{-4}{1-\tau^2} (\tau h_{\alpha\gamma} \delta_{\gamma\beta} - \tau h_{\beta\epsilon} \delta_{\alpha\epsilon}) = 0,\end{aligned}$$

where we use index ranges  $1 \leq \alpha, \beta, \gamma, \epsilon \leq k$ .  $\square$

### 3. THE AUSTERITY CONDITION

Let  $\mathbf{S}$  be the matrix on the right in (8). Then the pullback of the holomorphic volume form  $\Omega$  under  $\mathfrak{A} \circ \Phi \circ \iota$  equals  $\det \mathbf{S}$  times a real volume form on  $NM$ . In this section we will evaluate this determinant, and find conditions under which it has constant phase. In what follows, we again represent an arbitrary normal vector  $\nu \in NM$  by its horizontal lift in  $T_{\mathbf{z}} \widehat{M}$ , which is given by  $te_{2n}$  for some  $t \in \mathbb{R}$  and some adapted frame at  $\mathbf{z}$ .

First, we consider a special case. Assuming that  $\nu \in \mathcal{N}$ , then  $r_\alpha = 0$  for all  $\alpha$ , and we can factor  $\mathbf{S}$  as

$$\begin{pmatrix} I_k - \mathbf{i}\tau H & 0 & 0 \\ 0 & \mathbf{i}\tau I_{2n-k-1} & 0 \\ 0 & 0 & \mathbf{i}(1-\tau^2) \end{pmatrix} \begin{pmatrix} \check{e}_\alpha & ; & \check{e}_\alpha \\ \check{e}_\mu & ; & \check{e}_\mu \\ \mathbf{i}\check{E}_n & ; & -\mathbf{i}\check{E}_n \end{pmatrix}, \quad (13)$$

where  $I_k$  is the  $k \times k$  identity matrix and  $H$  is the matrix with entries  $h_{\alpha\beta}$ . Letting  $V$  be the matrix on the right in (13), we note that

$$\frac{1}{2}V \begin{pmatrix} I_n & -\mathbf{i}I_n \\ I_n & \mathbf{i}I_n \end{pmatrix} = \begin{pmatrix} \check{e}_\alpha \\ \check{e}_\mu \\ \mathbf{i}\check{E}_n \end{pmatrix}. \quad (14)$$

The matrix on the right-hand side of (14) lies in  $O(2n)$ , but it has determinant  $(-1)^n$ , since when we substitute our particular frame into (3), we obtain

$$1 = \det \begin{pmatrix} 1 & 0 \dots & 0 & 0 \dots \\ 0 & 0 \dots & 1 & 0 \dots \\ 0 & \operatorname{Re} \check{e}_\alpha & 0 & \operatorname{Im} \check{e}_\alpha \\ 0 & \operatorname{Re} \check{e}_\mu & 0 & \operatorname{Im} \check{e}_\mu \\ 0 & 0 \dots & 0 & \check{E}_n \end{pmatrix} = (-1)^n \det \begin{pmatrix} \operatorname{Re} \check{e}_\alpha & \operatorname{Im} \check{e}_\alpha \\ \operatorname{Re} \check{e}_\mu & \operatorname{Im} \check{e}_\mu \\ 0 \dots & \check{E}_n \end{pmatrix}.$$

Taking determinants on each side in (14) gives  $(\mathbf{i}/2)^n \det V = (-1)^n$ , so that  $\det V = (2\mathbf{i})^n$ . Thus,

$$\det \mathbf{S} = (-2)^n \mathbf{i}^{n-k} \tau^{2n-k-1} (1-\tau^2) \det(I_k - \mathbf{i}\tau H). \quad (15)$$



It is clear that the real part of  $i^{n-k} \det \mathbf{S}$  is nonzero for values of  $\tau$  in an open interval containing zero. On the other hand, by diagonalizing  $H$  it is easy to see that

$$\operatorname{Im} \det(I_k - i\tau H) = \sum_{j=1}^{\lfloor (k+1)/2 \rfloor} (-1)^j \tau^{2j-1} H^{(2j-1)}, \quad (16)$$

where  $H^{(2j-1)}$  denotes the elementary symmetric polynomial of degree  $2j-1$  in the eigenvalues of  $H$ . Thus, we conclude that

For  $\nu \in \mathcal{N}$ , the imaginary part of  $i^{n-k} \Omega$  vanishes along the line in  $NM$  spanned by  $\nu$  if and only if all odd-degree elementary symmetric polynomials in the eigenvalues of  $H$  vanish (where  $H$  represents  $\nu \cdot \Pi$  with respect to an orthonormal basis).

**Corollary 2.** *If  $M \subset \mathbb{C}P^n$  is a holomorphic submanifold, then  $M$  is austere.*

*Proof.* Because the complex structure  $J$  is parallel along  $M$ , the second fundamental form satisfies  $\Pi(X, JY) = J\Pi(X, Y) = \Pi(JX, Y)$ . If matrix  $J$  represents the complex structure with respect to an orthonormal basis, then  $HJ = J^T H = -JH$ , and hence  $JHJ = -J^2 H = H$ . So,

$$\det(I_k - i\tau H) = \det(I_k - i\tau J^{-1} H J) = \det(I_k + i\tau J H J) = \det(I_k + i\tau H),$$

and so  $\operatorname{Im} \det(I_k - i\tau H) = 0$ .  $\square$

Now consider the more general case, when  $\nu$  has a non-zero component in  $\mathcal{E}$ , and thus  $J\nu$  has a nonzero orthogonal projection onto  $\mathcal{D}$ . We will further specialize the orthonormal frame so that

$$ie_{2n} = \cos \theta e_1 + \sin \theta e_{2n-1},$$

where  $\theta$  is the angle between  $J\nu$  and the tangent space to  $M$ . Since we also have  $e_{2n} = iE_n$ , then  $r_1 = \cos \theta$ ,  $r_{2n-1} = \sin \theta$  and all other  $r_\alpha, r_\mu$  are zero.

To simplify calculating  $\det \mathbf{S}$ , we modify the matrix, adding  $\tau^2 r_1 / (\tau^2 - 1)$  times row  $2n$  to row  $1$ , giving

$$S' = \begin{pmatrix} \check{e}_\alpha - i\tau \check{H}_\alpha & ; & \check{\bar{e}}_\alpha - i\tau \check{\bar{h}}_\alpha - 2\tau^2 r_\alpha \check{E}_n \\ i\tau \check{e}_\mu & ; & i\tau \check{\bar{e}}_\mu \\ (\tau^2 - 1) \check{E}_n & ; & (1 - \tau^2) \check{E}_n \end{pmatrix},$$

where we introduce the abbreviation  $\check{H}_\alpha = h_{\alpha\beta} \check{e}_\beta$ . Again, only coefficient  $r_1 = \cos \theta$  is nonzero, and we expand the determinant in terms of it. Letting  $S_0$  denote the matrix in (13), we have

$$\det \mathbf{S} = \det S' = \det S_0 + (-1)^{2n-1} (-2\tau^2 \cos \theta) \det \begin{pmatrix} \check{e}_\beta - i\tau \check{H}_\beta & ; & \check{\bar{e}}_\beta - i\tau \check{\bar{h}}_\beta \\ i\tau \check{e}_\mu & ; & i\tau \check{\bar{e}}_\mu \\ (\tau^2 - 1) \check{E}_n & ; & 0 \end{pmatrix},$$

where we use the index range  $2 \leq \beta \leq k$ , and  $\sim$  indicates the result of deleting the first and last entries from a vector in  $\mathbb{C}^{n+1}$ . Thus, the matrix on the right is  $2n-1 \times 2n-1$ ; moreover, in the bottom row only the  $n$ th entry is nonzero, so we may use a cofactor expansion to write

$$\begin{aligned} \det \mathbf{S} &= \det S_0 + 2\tau^2 \cos \theta (-1)^{2n-2+n-1} (\tau^2 - 1) \det \begin{pmatrix} \widetilde{e}_\beta - i\tau \widetilde{H}_\beta & ; & \widetilde{\bar{e}}_\beta - i\tau \widetilde{\bar{h}}_\beta \\ i\tau \widetilde{e}_\mu & ; & i\tau \widetilde{\bar{e}}_\mu \end{pmatrix} \\ &= \det S_0 + 2 \cos \theta (-1)^{n-3} \tau^2 (\tau^2 - 1) \det \begin{pmatrix} I_{k-1} - i\tau \widetilde{H} & 0 \\ 0 & i\tau I_{2n-k-1} \end{pmatrix} \det \begin{pmatrix} \widetilde{e}_\beta & ; & \widetilde{\bar{e}}_\beta \\ \widetilde{e}_\mu & ; & \widetilde{\bar{e}}_\mu \end{pmatrix}, \end{aligned}$$

where  $\widetilde{H}$  is the  $(k-1) \times (k-1)$  matrix obtained from  $H$  by deleting the first row and column.

**Lemma 3.** *Let  $\widetilde{V} = \begin{pmatrix} \widetilde{e}_\beta & ; & \widetilde{\bar{e}}_\beta \\ \widetilde{e}_\mu & ; & \widetilde{\bar{e}}_\mu \end{pmatrix}$ . Then  $\det \widetilde{V} = (-2i)^{n-1} \cos \theta$ .*

Using the lemma (to be proved later), and the formula (15) for  $\det \mathbf{S}_0$ , we have

$$\begin{aligned} \det \mathbf{S} &= \det \mathbf{S}_0 + 2(-1)^{n-3} \cos^2 \theta \tau^2 (\tau^2 - 1) (i\tau)^{2n-k-1} (-2i)^{n-1} \det(I_{k-1} - i\tau \widetilde{H}) \\ &= (-2)^n \tau^{2n-k-1} [i^{n-k} (1 - \tau^2) \det(I_k - i\tau H) + i^{2n-k-1} (-i)^{n-1} \tau^2 (1 - \tau^2) \cos^2 \theta \det(I_{k-1} - i\tau \widetilde{H})] \\ &= 2^n i^{n-k} \tau^{2n-k-1} (1 - \tau^2) [\det(I_k - i\tau H) + \tau^2 \cos^2 \theta \det(I_{k-1} - i\tau \widetilde{H})]. \end{aligned}$$

As in (16),

$$\begin{aligned} \operatorname{Im} [\det(I_k - i\tau H) + \tau^2 \cos^2 \theta \det(I_{k-1} - i\tau \widetilde{H})] &= - \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} (-1)^j \tau^{2j+1} H^{(2j+1)} + \cos^2 \theta \sum_{j=1}^{\lfloor k/2 \rfloor} (-1)^j \tau^{2j+1} \widetilde{H}^{(2j-1)} \\ &= -\tau H^{(1)} + \tau^3 (H^{(3)} - \cos^2 \theta \widetilde{H}^{(1)}) - \tau^5 (H^{(5)} - \cos^2 \theta \widetilde{H}^{(3)}) \dots \end{aligned}$$

Thus, we conclude that

For  $\nu \notin \mathcal{N}$ , the imaginary part of  $i^{n-k} \Omega$  vanishes along the line in  $NM$  spanned by  $\nu$  if and only if

$$H^{(2j+1)} = \cos^2 \theta \widetilde{H}^{(2j-1)}, \quad 0 \leq j \leq \lfloor k/2 \rfloor, \quad (17)$$

where  $\theta$  is the angle between  $J\nu$  and the tangent space,  $\widetilde{H}$  denotes the restriction of  $\nu \cdot \Pi$  to the subspace of the tangent space orthogonal to  $J\nu$ , and the symmetric polynomials in (17) are understood to be zero if the degree is negative or larger than the size of the matrix.

Note that the previous condition, for  $\nu \in \mathcal{N}$ , is a special case of (17) for which  $\cos \theta = 0$ . We can therefore summarize our calculation as follows:

**Theorem 4.** *A submanifold  $M \subset \mathbb{C}P^n$  is austere if the condition (17) is satisfied for every normal vector  $\nu \in N_p M$  and every point  $p \in M$ .*

By setting  $j = 0$  in (17), we obtain:

**Corollary 5.** *If  $M \subset \mathbb{C}P^n$  is austere, then  $M$  is minimal.*

The following examples may help us understand the austerity condition:

**Example 1.** Assume  $M \subset \mathbb{C}P^n$  is a hypersurface. Then  $\mathcal{N}$  has rank zero,  $\mathcal{D}$  and  $\mathcal{E}$  have rank one, and  $J : \mathcal{D} \rightarrow \mathcal{E}$ . Hence  $\theta = 0$ , and we may write the austere conditions as

$$A^{(2j+1)} = \widetilde{A}^{(2j-1)}, \quad 0 \leq j \leq n-1,$$

where  $A$  denotes the scalar-valued second fundamental form of  $M$  and  $\widetilde{A}$  is its restriction to the holomorphic distribution  $\mathcal{H}$ .

**Example 2.** Assume  $M \subset \mathbb{C}P^n$  is a curve. Then  $M$  is austere if and only if it is a geodesic.

Note that when  $k = \dim M$  is even and  $\cos \theta \neq 0$ , the last condition in (17), for  $j = k/2$ , gives  $\widetilde{H}^{(k-1)} = 0$ .

**Example 3.** Assume  $M \subset \mathbb{C}P^n$  is a surface which is not holomorphic. Hence,  $\mathcal{H}$  has rank zero. Then  $M$  is austere if and only if  $M$  is minimal and the ‘last condition’, obtained by setting  $j = 1$  in (17) holds; this condition is that  $\Pi(\mathbf{v}, \mathbf{v}) = 0$ , where  $\mathbf{v} \in T_p M$  is the vector orthogonal to  $J\nu$ , and  $\nu$  runs over the unit circle bundle in  $\mathcal{E}_p$  for all  $p \in M$ .

Austere surfaces are discussed in more detail in the next section.

*Proof of Lemma 3.* As observed after equation (14),

$$(-1)^n = \det \begin{pmatrix} \widetilde{e_\alpha} \\ \widetilde{e_\mu} \\ \widetilde{iE_n} \end{pmatrix} = \det \begin{pmatrix} \operatorname{Re} \widetilde{e_1} & -\cos \theta & \operatorname{Im} \widetilde{e_1} & 0 \\ \operatorname{Re} \widetilde{e_\beta} & 0 & \operatorname{Im} \widetilde{e_\beta} & 0 \\ \operatorname{Re} \widetilde{e_\lambda} & 0 & \operatorname{Im} \widetilde{e_\lambda} & 0 \\ \operatorname{Re} \widetilde{e_{2n-1}} & -\sin \theta & \operatorname{Im} \widetilde{e_{2n-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where we take index ranges  $1 < \beta \leq k$  and  $k < \lambda < 2n - 1$ . Because the first and last entries of  $e_\beta$  and  $e_\lambda$  are zero, vectors  $\widetilde{e_\beta}$  and  $\widetilde{e_\lambda}$  are mutually orthogonal unit vectors in  $\mathbb{C}^{n-1}$ . Since  $\widetilde{e_1}$  and  $\widetilde{e_{2n-1}}$  are orthogonal to all of them, these two vectors must be linearly dependent over  $\mathbb{R}$ . Since  $\widetilde{e_{2n-1}}$  is a unit vector with  $\widetilde{e_{2n-1}} \cdot \widetilde{E_n} = -\sin \theta$ , then  $|\widetilde{e_{2n-1}}|^2 = \cos^2 \theta \neq 0$ . If we set  $\widetilde{e_1} = a \widetilde{e_{2n-1}}$  for a scalar  $a$ , then solving  $0 = \langle \widetilde{e_1}, \widetilde{e_{2n-1}} \rangle$  gives  $a = -\tan \theta$ . Thus, adding  $\tan \theta$  times the second-last row to the first row in the matrix gives

$$(-1)^n = \det \begin{pmatrix} 0 & -\sec \theta & 0 & 0 \\ \operatorname{Re} \widetilde{e_\beta} & 0 & \operatorname{Im} \widetilde{e_\beta} & 0 \\ \operatorname{Re} \widetilde{e_\lambda} & 0 & \operatorname{Im} \widetilde{e_\lambda} & 0 \\ \operatorname{Re} \widetilde{e_{2n-1}} & -\sin \theta & \operatorname{Im} \widetilde{e_{2n-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (-1)^n \sec \theta \det \begin{pmatrix} \widetilde{e_\beta} \\ \widetilde{e_\lambda} \\ \widetilde{e_{2n-1}} \end{pmatrix}.$$

On the other hand, as in (14) we have

$$\frac{1}{2} \widetilde{V} \begin{pmatrix} I_{n-1} & -iI_{n-1} \\ I_{n-1} & iI_{n-1} \end{pmatrix} = \begin{pmatrix} \widetilde{e_\beta} \\ \widetilde{e_\lambda} \\ \widetilde{e_{2n-1}} \end{pmatrix}.$$

Taking determinants on each side and solving gives the desired formula for  $\det \widetilde{V}$ .  $\square$

#### 4. CLASSIFICATION OF AUSTERE SURFACES

In this section we classify surfaces in  $\mathbb{C}P^n$  that satisfy the austere condition of Theorem 4.

**Proposition 6.** *Let  $M \subset \mathbb{C}P^n$  be an austere surface such that  $\mathcal{H} = 0$  at every point. Then  $M$  is totally geodesic.*

*Proof.* By assumption, the splitting (1) implies  $TM = \mathcal{D}$ ,  $NM = \mathcal{E} \oplus \mathcal{N}$ , where  $\mathcal{D}, \mathcal{E}$  are rank 2 and  $\mathcal{D} \oplus \mathcal{E}$  is  $J$ -invariant. Let  $p \in M$  be an arbitrary point and let  $\nu \in N_p M$  be an arbitrary unit normal vector. At  $p$  we will construct a orthonormal basis for  $T_p \mathbb{C}P^n$  of the form  $(e_1, Je_1, e_2, Je_2, \dots, e_n, Je_n)$ , which we will refer to as a *unitary frame*.

Fix a unit vector  $e_1 \in \mathcal{D}_p$ , and let  $\theta$  be the angle between  $Je_1$  and  $\mathcal{D}_p$ . This is the *Kähler angle*, which is independent of the choice of  $e_1$  and nonzero by assumption. Thus,  $Je_1$  has a nonzero orthogonal projection onto  $\mathcal{E}_p$ ; let  $w$  be the unit vector in the direction opposite to this projection, and let  $e_2 \in \mathcal{E}_p$  be a choice of unit vector orthogonal to  $w$ . Then  $e_1, Je_1, e_2$  are linearly independent, and thus  $(e_1, Je_1, e_2, Je_2)$  is a basis for  $\mathcal{D}_p \oplus \mathcal{E}_p$ . Since  $\langle w, Je_1 \rangle = -\sin \theta$ , we must have  $w = -\sin \theta Je_1 \pm \cos \theta Je_2$ . We adjust the sense of  $e_2$  so that

$$w = -\sin \theta Je_1 + \cos \theta Je_2. \quad (18)$$

Then

$$v = \cos \theta Je_1 + \sin \theta Je_2$$

is orthogonal to  $w$  and  $e_2$ , and thus  $(e_1, v)$  is an orthonormal basis for  $\mathcal{D}_p = T_p M$ .

We now choose the remaining vectors of the unitary frame so that  $e_3$  is the unit vector in the direction of the orthogonal projection of  $\nu$  onto  $\mathcal{N}_p$ , and  $\mathcal{N}_p$  is spanned by  $e_3, Je_3, \dots, e_n, Je_n$ . Let  $\psi$  be the angle between  $\nu$  and  $\mathcal{N}_p$ , and let  $II'$  be the quadratic form on  $T_p M$  given by  $\nu \cdot II$ . If  $\psi = 0$

(i.e.,  $\nu \in \mathcal{N}_p$ ) then the austere condition reduces to  $\text{tr } \Pi^\nu = 0$ . We are interested in what additional conditions arise when  $\psi \neq 0$ , so we assume this from now on.

Let  $\mathbf{u}$  be the unit vector in the direction of the orthogonal projection of  $\nu$  onto  $\mathcal{E}_p$ . Then

$$\nu = \cos \psi \mathbf{e}_3 + \sin \psi \mathbf{u}. \quad (19)$$

Let  $\varphi$  be the angle such that  $\mathbf{u} = \cos \varphi \mathbf{e}_2 + \sin \varphi \mathbf{w}$ . When we substitute this, and then (18), into (19) we obtain

$$\nu = \cos \psi \mathbf{e}_3 + \sin \psi (\cos \varphi \mathbf{e}_2 + \sin \varphi \mathbf{w}) \quad (20)$$

$$= \cos \psi \mathbf{e}_3 + \sin \psi (\cos \varphi \mathbf{e}_2 - \sin \varphi \sin \theta J\mathbf{e}_1 + \sin \varphi \cos \theta J\mathbf{e}_2). \quad (21)$$

It is important to note that, while fixing  $\mathbf{e}_3$ , we can vary the normal vector  $\nu$  by varying the angles  $\psi$  and  $\varphi$  independently.

As in Example 3, the austere condition requires that  $M$  be minimal and that  $\Pi^\nu(\xi, \xi) = 0$ , where  $\xi \in T_p M$  is a unit vector orthogonal to  $J\nu$ . By computing  $J\nu$  using (21), one can verify that

$$\xi = \cos \varphi \mathbf{e}_1 - \sin \varphi \mathbf{v}. \quad (22)$$

To compute  $\Pi^\nu$ , we express the second fundamental form in our basis normal directions as

$$\Pi^{\mathbf{w}} = \begin{pmatrix} a_1 & b_1 \\ b_1 & -a_1 \end{pmatrix}, \quad \Pi^{\mathbf{e}_2} = \begin{pmatrix} a_2 & b_2 \\ b_2 & -a_2 \end{pmatrix}, \quad \Pi^{\mathbf{e}_3} = \begin{pmatrix} a_3 & b_3 \\ b_3 & -a_3 \end{pmatrix},$$

where each quadratic form is represented by a matrix with respect to the orthonormal basis  $(\mathbf{e}_1, \mathbf{v})$ .

Now using (20) and (22), the condition  $\Pi^\nu(\xi, \xi) = 0$  can be expanded as

$$\begin{bmatrix} \cos \varphi & -\sin \varphi \end{bmatrix} (\sin \psi \sin \varphi \Pi^{\mathbf{w}} + \sin \psi \cos \varphi \Pi^{\mathbf{e}_2} + \cos \psi \Pi^{\mathbf{e}_3}) \begin{bmatrix} \cos \varphi \\ -\sin \varphi \end{bmatrix}.$$

Since this equation must be satisfied for all angles  $\varphi$  and  $\psi$  (provided  $\sin \psi \neq 0$ ), we can take particular values. Setting  $\varphi = 0$  yields

$$a_2 \sin \psi + a_3 \cos \psi = 0,$$

which can only hold for all  $\psi$  if  $a_2 = a_3 = 0$ . Setting  $\varphi = \pi/2$  yields

$$a_1 \sin \psi + a_3 \cos \psi = 0,$$

so that  $a_1$  must also vanish. Taking  $a_1 = a_2 = a_3 = 0$  into account, the condition becomes

$$\sin \varphi \cos \varphi (\sin \psi (b_1 \sin \varphi + b_2 \cos \varphi) + b_3 \cos \psi) = 0,$$

while implies that  $b_1 = b_2 = b_3 = 0$ .

Since  $\nu$  is an arbitrary normal direction,  $\mathbf{e}_3$  can range over all of  $\mathcal{N}_p$ , and  $p$  is arbitrary, we conclude that  $M$  is totally geodesic.  $\square$

**Theorem 7.** *If  $M \subset \mathbb{C}P^n$  is a connected austere surface, then  $M$  is either holomorphic or an open subset of a real projective plane  $\mathbb{R}P^2 \subset \mathbb{C}P^2$ , where  $\mathbb{C}P^2$  is embedded as a complex linear subspace in  $\mathbb{C}P^n$ .*

*Proof.* If  $M$  is austere, then  $M$  is minimal and hence real-analytic. Thus, the points where  $T_p M$  is  $J$ -invariant form an open and closed subset of  $M$ . By Proposition 6,  $M$  is either holomorphic or totally geodesic. If  $M$  is totally geodesic, then by a well-known result of Wolf [12],  $M$  is either an open set of a complex line in  $\mathbb{C}P^n$ , or the real part of a complex projective plane.  $\square$

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